

Spectral Bounds for the Connectivity of Regular Graphs with Given Order

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Abstract

The second-largest eigenvalue and second-smallest Laplacian eigenvalue of a graph are measures of the graph's connectivity. These parameters can be used to analyze the robustness, resilience, and synchronizability of networks, and are related to other connectivity attributes such as the vertex- and edge-connectivity, isoperimetric number, and characteristic path length. In this paper, we give upper bounds for the second-largest eigenvalues of regular graphs and multigraphs which guarantee a desired vertex- or edge-connectivity. The given bounds are in terms of the order and degree of the graphs, and hold with equality for infinite families of graphs.

Keywords. Second-largest eigenvalue; vertex-connectivity; edge-connectivity; regular multigraph; algebraic connectivity.

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1 Introduction

Determining how well a particular graph is connected is a problem that arises often in various applications. For example, as defined in [22], *resilience* is the ability of a network to provide and maintain an acceptable level of service in the face of various faults and challenges to normal operation. Examples of such networks include telecommunication networks, electrical grids, and commodity supply networks. One of the scientific disciplines that forms the basis of resilience is *disruption tolerance* – the ability of a system to tolerate disruptions in connectivity among its components; such disruptions could consist of environmental challenges such as weak and episodic channel connectivity, mobility, and unpredictably-long delay, as well as tolerance of energy challenges [22]. The most common way to interpret such networks is as connected graphs, where the resilience of a network as a function of its disruption tolerance can be studied by analyzing the connectivity properties of the associated graph. Graph connectivity has also been shown to play an important role in data security; see [14] and the references therein for more information.

In 1973, Fiedler related the vertex-connectivity of a graph to the second-smallest eigenvalue of its Laplacian matrix in the following way (see Section 2 for relevant definitions):

Theorem 1.1. [6] *If G is a simple, non-complete graph, then $\kappa(G) \geq \mu_2(G)$.*

Note that Theorem 1.1 also implies $\kappa'(G) \geq \mu_2(G)$, since $\kappa'(G) \geq \kappa(G)$. This seminal result provided researchers with another parameter that quantitatively measures the connectivity of a graph; hence, $\mu_2(G)$ is known as the *algebraic connectivity* of G . Fiedler’s discovery ignited interest in studying the connectivity of graphs by analyzing the spectral properties of their associated matrices. Akin to other connectivity measures such as vertex-connectivity, edge-connectivity, and isoperimetric number, the algebraic connectivity of a graph has applications in the design of reliable communication networks [17] and in analyzing the robustness of complex networks [12, 13].

Recall that for a d -regular multigraph G on n vertices, $\lambda_i(G) = d - \mu_i(G)$ for $i = 1, \dots, n$. Thus, for regular multigraphs, spectral bounds related to connectivity are often expressed in terms of the second-largest eigenvalue, instead of the second-smallest Laplacian eigenvalue. We now discuss several results relating the vertex- and edge-connectivity of graphs to their second-largest eigenvalues, which are similar in nature to the main results of the

present paper.

Literature review

The following result of Chandran [3] relates the second-largest eigenvalue of an n -vertex d -regular graph to its edge-connectivity.

Theorem 1.2. [3] *Let G be an n -vertex d -regular simple graph with $\lambda_2(G) < d - 1 - \frac{d}{n-d}$. Then $\kappa'(G) = d$.*

In 2006, Krivelevich and Sudakov [16] improved the above result as follows.

Theorem 1.3. [16] *Let G be a d -regular simple graph with $\lambda_2(G) \leq d - 2$. Then $\kappa'(G) = d$.*

The next result was shown in 2010 by Ciobă [5].

Theorem 1.4. [5] *Let t be a nonnegative integer less than d , and let G be a d -regular, simple graph with $\lambda_2(G) < d - \frac{2t}{d+1}$. Then $\kappa'(G) \geq t + 1$.*

In the same paper, Ciobă also gave improvements of Theorem 1.4 for the following two particular cases.

Theorem 1.5. [5] *Let $d \geq 3$ be an odd integer and let $\pi(d)$ denote the largest root of $x^3 - (d-3)x^2 - (3d-2)x - 2 = 0$. If G is a d -regular, simple graph such that $\lambda_2(G) < \pi(d)$, then $\kappa'(G) \geq 2$.*

The value of $\pi(d)$ above is approximately $d - \frac{2}{d+5}$. For $t \geq 3$, it is still an open problem to find the best upper bound for $\lambda_2(G)$ in a d -regular simple graph G to guarantee that $\kappa(G) \geq t + 1$.

Theorem 1.6. [5] *Let $d \geq 3$ be any integer. Let G be a d -regular, simple graph with*

$$\lambda_2(G) < \frac{d-3 + \sqrt{(d+3)^2 - 16}}{2}.$$

Then $\kappa'(G) \geq 3$.

The value of $\frac{d-3 + \sqrt{(d+3)^2 - 16}}{2}$ above is approximately $d - \frac{4}{d+3}$.

In 2016, O [19] generalized Fiedler's result to multigraphs, and established similar bounds as those above.

Theorem 1.7. [19] *Let G be a connected, d -regular multigraph with*

$$\lambda_2(G) < \frac{d - 1 + \sqrt{9d^2 - 10d + 17}}{4}.$$

Then $\kappa'(G) \geq 2$.

Theorem 1.8. [19] *Let $t \geq 2$ and let G be a connected, d -regular multigraph. If $\lambda_2(G) < d - t$, then $\kappa'(G) \geq t + 1$. Furthermore, if t is odd and $\lambda_2(G) < d - t + 1$, then $\kappa'(G) \geq t + 1$.*

Moreover, for every positive integer t less than d , O [19] found the best upper bound for $\lambda_2(G)$ to guarantee that $\kappa'(G) \geq t + 1$.

Note that the bounds for $\lambda_2(G)$ in Theorems 1.5 and 1.6 are larger than those in Theorems 1.7 and 1.8; this suggests that if multiple edges are not allowed, then a bound for $\lambda_2(G)$ that will guarantee a desired edge-connectivity may be larger. Hence, we see that a d -regular multigraph G may have high edge-connectivity while $\lambda_2(G)$ is small; this supports the implicit meaning of Cheeger's inequality (see [10, p. 454]) for a d -regular multigraph G , which asserts that

$$\frac{d - \lambda_2(G)}{2} \leq h(G) \leq \sqrt{2d(d - \lambda_2(G))},$$

where

$$h(G) = \min_{\substack{S \subseteq V(G), \\ 1 \leq |S| \leq \frac{n}{2}}} \frac{[S, \bar{S}]}{|S|}.$$

The Cheeger constant $h(G)$ (also known as the isoperimetric number) can be considered as an approximation to the edge-connectivity of G .

The results above make assertions about the edge-connectivity of a graph based on its eigenvalues. In a more recent paper, O [18] also established analogous results for vertex-connectivity.

Theorem 1.9. [18] *Let G be a d -regular multigraph that is not the 2-vertex d -regular multigraph. If $\lambda_2(G) < \frac{3d}{4}$, then $\kappa(G) \geq 2$.*

Algebraic connectivity has also been studied in the context of hypergraphs [11] and directed graphs [25]. See [1, 4, 7, 15, 20] and the bibliographies therein for other recent results on algebraic connectivity, more general structural results on vertex- and edge-connectivity, and other algebraic measures of connectivity.

Main contributions

The aim of the present paper is to investigate what upper bounds on the second-largest eigenvalues of regular simple graphs and multigraphs guarantee a desired vertex- or edge-connectivity. In other words, we address the following question:

Question 1.10. *For a d -regular simple graph or multigraph G and for $1 \leq t \leq d - 1$, what is the best upper bound for $\lambda_2(G)$ which guarantees that $\kappa'(G) \geq t + 1$ or that $\kappa(G) \geq t + 1$?*

The majority of the related results listed above were derived using a variety of combinatorial, linear algebraic, and analytic techniques; moreover, they feature upper bounds for $\lambda_2(G)$ which do not depend on the order of the graph. In contrast, the results derived in the present paper feature bounds for $\lambda_2(G)$ which depend on both the degree and the order of the graphs, and as such are tight for infinite families of graphs. Furthermore, the derivations of these results combine analytic techniques with computer-aided *symbolic* algebra; this proves to be a powerful approach, easily establishing the desired results in all but finitely-many cases. The remaining cases are verified through a brute-force approach which relies on enumerating all multigraphs with certain properties. In order to avoid enumeration and post-hoc elimination of the exponential number of multigraphs without the desired properties, our approach required the development of novel combinatorial and graph theoretic techniques. While the problem of generating all non-isomorphic simple graphs having a certain degree sequence and other properties is well-studied (cf. [8, 9, 21]), there are not as many efficiently-implemented algorithms for constrained enumeration of multigraphs (see [23] for some results in this direction). Thus, the developed enumeration procedure may also be of independent interest.

The paper is organized as follows. In the next section, we recall some graph theoretic and linear algebraic notions, specifically those related to eigenvalue interlacing. In Sections 3 and 4, respectively, we give spectral bounds which guarantee a certain vertex- and edge-connectivity. We conclude with some final remarks and open questions in Section 5. The Appendix includes further details and computer code for symbolic computations used in some of the proofs.

2 Preliminaries

In this paper, a *multigraph* refers to a graph with multiple edges but no loops; a *simple graph* refers to a graph with no multiple edges or loops. The *order* and *size* of a multigraph G are denoted by $n = |V(G)|$ and $m = |E(G)|$, respectively. A *double edge* (respectively *triple edge*) in a multigraph is an edge of multiplicity two (respectively three). The *degree* of a vertex v of G , denoted $d_G(v)$, is the number of edges incident to v . The *degree sequence* of G is a list $\{d_1, \dots, d_n\}$ of the vertex degrees of G . We may abbreviate the degree sequence of G by only writing distinct degrees, with the number of vertices realizing each degree in superscript. For example, if G is the star graph on n vertices, the degree sequence of G may be written as $\{n-1, 1^{n-1}\}$.

A *vertex cut* (respectively *edge cut*) of G is a set of vertices (respectively edges) which, when removed, increases the number of connected components in G . A multigraph G with more than k vertices is said to be *k -vertex-connected* if there is no vertex cut of size $k-1$. The *vertex-connectivity* of G , denoted $\kappa(G)$, is the maximum k such that G is k -vertex-connected. Similarly, G is *k -edge-connected* if there is no edge cut of size $k-1$; the *edge-connectivity* of G , denoted $\kappa'(G)$, is the maximum k such that G is k -edge-connected. A *cut-vertex* (respectively *cut-edge*) is a vertex cut (respectively edge cut) of size one.

Given sets $V_1, V_2 \subset V(G)$, $[V_1, V_2]$ denotes the number of edges with one endpoint in V_1 and the other in V_2 . The *induced subgraph* $G[V_1]$ is the subgraph of G whose vertex set is V_1 and whose edge set consists of all edges of G which have both endpoints in V_1 . A *matching* is a set of edges of G which have no common endpoints; a *k -matching* is a matching containing k edges. $G + e$ denotes the graph $(V(G), E(G) \cup \{e\})$, and $G + E'$ denotes the graph $(V(G), E(G) \cup E')$. The *complete graph* on n vertices is denoted K_n . An *odd path* (respectively *even path*) in a graph is a connected component which is a path with an odd (respectively even) number of vertices. For other graph theoretic terminology and definitions, we refer the reader to [24].

The *adjacency matrix* of G will be denoted by $A(G)$; recall that in a multigraph, the entry $A_{i,j}$ is the number of edges between vertices v_i and v_j . The *eigenvalues* of G are the eigenvalues of its adjacency matrix, and are denoted by $\lambda_1(G) \geq \dots \geq \lambda_n(G)$. The *Laplacian matrix* of G is equal to $D(G) - A(G)$, where $D(G)$ is the diagonal matrix whose entry $D_{i,i}$ is the degree of vertex v_i . The *Laplacian eigenvalues* of G are the eigenvalues of its Laplacian matrix and are denoted by $0 = \mu_1(G) \leq \dots \leq \mu_n(G)$. The

dependence of these parameters on G may be omitted when it is clear from the context.

A technical tool used in this paper is *eigenvalue interlacing* (for more details see Section 2.5 of [2]). Given two sequences of real numbers $a_1 \geq \dots \geq a_n$ and $b_1 \geq \dots \geq b_m$ with $m < n$, we say that the second sequence *interlaces* the first sequence whenever $a_i \geq b_i \geq a_{n-m+i}$ for $i = 1 \dots m$.

Theorem 2.1. [Interlacing Theorem, [2]] *If A is a real symmetric $n \times n$ matrix and B is a principal submatrix of A of order $m \times m$ with $m < n$, then for $1 \leq i \leq m$, $\lambda_i(A) \geq \lambda_i(B) \geq \lambda_{n-m+i}(A)$, i.e., the eigenvalues of B interlace the eigenvalues of A .*

Let $\mathcal{P} = \{V_1, \dots, V_s\}$ be a partition of the vertex set of a multigraph G into s non-empty subsets. The *quotient matrix* Q corresponding to \mathcal{P} is the $s \times s$ matrix whose entry $Q_{i,j}$ ($1 \leq i, j \leq s$) is the average number of incident edges in V_j of the vertices in V_i . More precisely, $Q_{i,j} = \frac{|V_i, V_j|}{|V_i|}$ if $i \neq j$, and $Q_{i,i} = \frac{2|E(G[V_i])|}{|V_i|}$. Note that for a simple graph, $Q_{i,j}$ is just the average number of neighbors between vertices in V_j and vertices in V_i .

Corollary 2.2. [Corollary 2.5.4, [2]] *The eigenvalues of any quotient matrix Q interlace the eigenvalues of G .*

3 Bounds for $\lambda_2(G)$ to guarantee $\kappa(G) \geq t + 1$

3.1 $\lambda_2(G)$ and $\kappa(G) \geq t + 1$

In this section, we establish an upper bound for the second-largest eigenvalue of an n -vertex d -regular simple graph or multigraph which guarantees a certain vertex-connectivity. To our knowledge, this is the first spectral bound on the vertex-connectivity of a regular graph which depends on both the degree and the order of the graph.

Theorem 3.1. *Let G be an n -vertex d -regular simple graph or multigraph, which is not obtained by duplicating edges in a complete graph on at most $t + 1$ vertices; let*

$$\phi(d, t) = \begin{cases} 3 & \text{if } G \text{ is a multigraph and } t = 1 \\ t + 1 & \text{if } G \text{ is a multigraph and } t \geq 2 \\ d + 2 & \text{if } G \text{ is a simple graph and } t = 1 \\ d + 1 & \text{if } G \text{ is a simple graph and } t \geq 2, \end{cases}$$

where $0 \leq t \leq d - 1$. If $\lambda_2(G) < d - \frac{td}{2\phi(d,t)} - \frac{td}{2(n-\phi(d,t))}$, then $\kappa(G) \geq t + 1$.

Proof. Assume to the contrary that $\kappa(G) \leq t$. If G is disconnected, then $\lambda_2(G) = d \geq d - \frac{td}{2\phi(d,t)} - \frac{td}{2(n-\phi(d,t))}$, a contradiction. Now, assume that $\kappa(G) \geq 1$. Hence, there exists a vertex cut C of G with $1 \leq c := |C| \leq t$. Let S_1 be a union of some components of $G - C$ such that $[S, \bar{S}] = [C, \bar{S}] \leq \frac{cd}{2} \leq \frac{td}{2}$, where $S = S_1 \cup C$ and $\bar{S} = V(G) \setminus S$. See Figure 1 for an illustration of this partition.

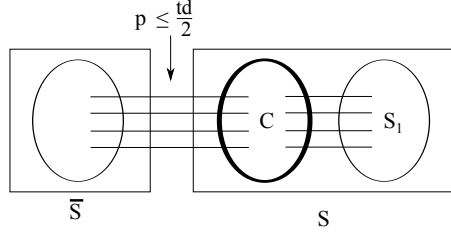


Figure 1: Partition of $V(G)$ into S and \bar{S} .

Let $[S, \bar{S}] = p$, and $|S_1| = s_1$; then, we have $2[S, S] = d(s_1 + c) - p$, and $2[\bar{S}, \bar{S}] = d(n - s_1 - c) - p$, so the quotient matrix for the partition $\{S, \bar{S}\}$ is

$$Q = \begin{pmatrix} d - \frac{p}{s_1 + c} & \frac{p}{s_1 + c} \\ \frac{p}{n - s_1 - c} & d - \frac{p}{n - s_1 - c} \end{pmatrix},$$

and the characteristic polynomial of Q with respect to x is $(x - d)(x - d + \frac{p}{s_1 + c} + \frac{p}{n - s_1 - c})$. Then by Corollary 2.2, we have

$$\lambda_2(G) \geq d - \frac{p}{s_1 + c} - \frac{p}{n - s_1 - c}.$$

We now consider two cases based on whether G is a simple graph or a multigraph.

Case 1: G is a simple graph. If $t = 1$, since the degree of each vertex in S_1 is d , it holds that $s_1 \geq d$. If $s_1 = d$, $G[S]$ is a complete subgraph of G , so the vertex in C has degree greater than d because $p \geq 1$; this is a contradiction. Thus $s_1 \geq d + 1$, $p \leq \frac{d}{2}$, and so $\lambda_2(G) \geq d - \frac{d}{2(d+2)} - \frac{d}{2(n-d-2)}$, as desired. If $t \geq 2$, by the same argument as above, we must have $s_1 \geq d + 1 - c$, $p \leq \frac{td}{2}$, and so $\lambda_2(G) \geq d - \frac{td}{2(d+1)} - \frac{td}{2(n-d-1)}$, as desired.

Case 2: G is a multigraph. If $t = 1$, then $s_1 \geq 2$, $p \leq d/2$, and so $\lambda_2(G) \geq d - \frac{d}{6} - \frac{d}{2(n-3)}$, as desired. If $t \geq 2$, then $s_1 \geq 1$, $p \leq \frac{cd}{2}$, and so $\lambda_2(G) \geq d - \frac{cd}{2(c+1)} - \frac{cd}{2(n-c-1)}$. Consider the function $f(c) = d - \frac{cd}{2(c+1)} - \frac{cd}{2(n-c-1)}$; then, $f'(c) = -\frac{d}{2} \left[\frac{1}{(c+1)^2} + \frac{n-1}{(n-c-1)^2} \right] < 0$. Thus, $f(c)$ is decreasing with respect to c for $1 \leq c \leq t$, whence it follows that $\lambda_2(G) \geq d - \frac{td}{2(t+1)} - \frac{td}{2(n-t-1)}$, as desired. \square

3.2 Improved bound for $\lambda_2(G)$ to guarantee $\kappa(G) \geq 2$

We now improve the result of Theorem 3.1 in the case when G is a multigraph and $t = 1$. Recall that in this case, Theorem 3.1 states that if $\lambda_2(G) < d - \frac{d}{6} - \frac{d}{2(n-3)} = \frac{5n-18}{6n-18}d$, then $\kappa(G) \geq 2$.

Theorem 3.2. *Let G be an n -vertex d -regular multigraph with $n \geq 5$ and $d \geq 3$. If $\lambda_2(G) < \frac{8n-25}{9n-25}d$, then $\kappa(G) \geq 2$.*

Proof. Assume to the contrary that $\kappa(G) \leq 1$. If $\kappa(G) = 0$, then $\lambda_2(G) = d > \frac{8n-25}{9n-25}d$, a contradiction. Thus, we can assume henceforth that $\kappa(G) = 1$.

Let v be a cut-vertex of G , and S_1 and S_2 be two components of $G - v$ with $|S_1| = s_1$ and $|S_2| = s_2 = n - s_1 - 1$. Let $m_1 = [v, S_1]$ and $m_2 = [v, S_2]$; without loss of generality, we can assume that $m_2 \leq m_1$, and hence that $1 \leq m_2 \leq \frac{d}{2}$ (otherwise the roles of S_1 and S_2 can be reversed); note that since $d \geq 3$, we must have $2 \leq s_1 \leq n - 3$; moreover, $d = m_1 + m_2$. See Figure 2 for an illustration of this partition in the case when $s_1 = 2$.

The quotient matrix for the partition $\{S_1, \{v\}, S_2\}$ is

$$Q = \begin{pmatrix} d - \frac{m_1}{s_1} & \frac{m_1}{s_1} & 0 \\ m_1 & 0 & m_2 \\ 0 & \frac{m_2}{s_2} & d - \frac{m_2}{s_2} \end{pmatrix},$$

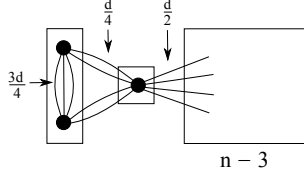


Figure 2: Partition of $V(G)$ into S_1 , $\{v\}$ and S_2 , when $|S_1| = 2$.

and its characteristic polynomial with respect to x is

$$(x - d) \left[x^2 - \left(d - \frac{m_1}{s_1} - \frac{m_2}{s_2} \right) x - \frac{m_1^2}{s_1} - \frac{m_2^2}{s_2} + \frac{m_1 m_2}{s_1 s_2} \right].$$

Then by Corollary 2.2, we have $\lambda_2(G) \geq \lambda_2(Q)$, where $\lambda_2(Q)$ is the second-largest root of the characteristic polynomial of Q ; it can be verified that $\lambda_2(Q)$ can be expressed as follows:

$$\frac{1}{2} \left[d - \frac{m_1}{s_1} - \frac{m_2}{s_2} + \sqrt{\left(d - \frac{m_1}{s_1} - \frac{m_2}{s_2} \right)^2 + 4 \left(\frac{m_1^2}{s_1} + \frac{m_2^2}{s_2} - \frac{m_1 m_2}{s_1 s_2} \right)} \right]. \quad (1)$$

If we set the derivative of $\lambda_2(Q)$ with respect to m_2 equal to zero and solve for m_2 , we obtain

$$m_2 = \frac{d(s_2 + 2s_1 s_2)}{n - 1 + 4s_1 s_2}. \quad (2)$$

Substituting $d - m_2$ for m_1 , and the right hand side of (2) for m_2 in (1), and simplifying, we obtain

$$\lambda_2(G) \geq d - \frac{dn}{n - 1 + 4s_1 s_2}.$$

Finally, when we substitute $n - s_1 - 1$ for s_2 , the resulting expression has a minimum at $s_1 = 2$, for $n \geq 5$, $d \geq 3$, and $2 \leq s_1 \leq n - 3$, with minimal value $\frac{8dn - 25d}{9n - 25}$. This minimization and some of the algebraic manipulations described above were carried out using symbolic computation in Mathematica; for details, see the Appendix. \square

Observation 3.3. *Let G be a multigraph with the following adjacency matrix:*

$$\begin{pmatrix} 0 & 3d/4 & d/4 & 0 & 0 \\ 3d/4 & 0 & d/4 & 0 & 0 \\ d/4 & d/4 & 0 & d/4 & d/4 \\ 0 & 0 & d/4 & 0 & 3d/4 \\ 0 & 0 & d/4 & 3d/4 & 0 \end{pmatrix}.$$

Then $\lambda_2(G) = \frac{8 \cdot 5 - 25}{9 \cdot 5 - 25}d$. Moreover, G is a d -regular multigraph with 5 vertices, $d = 4k$, $k \geq 1$, and $\kappa(G) = 1$. Thus, the bound in Theorem 3.2 is the best possible for this infinite family of multigraphs.

4 Bounds for $\lambda_2(G)$ to guarantee $\kappa'(G) \geq t + 1$

In this section, we first give an upper bound for $\lambda_2(G)$ in an n -vertex d -regular multigraph which guarantees that $\kappa'(G) \geq t + 1$; its proof is omitted, since it is similar to that of Theorem 3.1. Theorem 4.1 extends a result of Ciobă [5] to multigraphs.

Theorem 4.1. *Let G be an n -vertex d -regular multigraph, which is not obtained by duplicating edges in a complete graph on at most $t + 1$ vertices. Let*

$$\psi(d, t) = \begin{cases} 3 & \text{if } t = 1 \\ 2 & \text{if } t \geq 2, \end{cases}$$

where $0 \leq t \leq d - 1$. If $\lambda_2(G) < d - \frac{t}{\psi(d, t)} - \frac{t}{n - \psi(d, t)}$, then $\kappa'(G) \geq t + 1$.

Now, we will improve the bound in Theorem 4.1 for the case of $t = 1$; see Observation 4.3 for an explanation of why Theorem 4.2 is an improvement.

Theorem 4.2. *Let G be an n -vertex d -regular multigraph with $\lambda_2(G) < \rho(d, n)$, where $\rho(d, n)$ is the second-largest eigenvalue of the following matrix:*

$$Q = \begin{pmatrix} \frac{d+1}{2} & \frac{d-1}{2} & 0 & 0 \\ d-1 & 0 & 1 & 0 \\ 0 & 1 & 0 & d-1 \\ 0 & 0 & \frac{d-1}{n-4} & d - \frac{d-1}{n-4} \end{pmatrix}.$$

Then $\kappa'(G) \geq 2$.

Proof. Assume to the contrary that $\kappa'(G) \leq 1$. If $\kappa'(G) = 0$, then since the largest eigenvalue of Q equals d , we have that $\lambda_2(G) = d = \lambda_1(Q) \geq \lambda_2(Q) = \rho(d, n)$, a contradiction.

Now, assume that $\kappa'(G) = 1$. For any graph H , define $sc(H)$ to be the number of vertices in the smallest connected component of H . Let $e = v_1v_2$ be a cut-edge of G such that $sc(G - e) = \min\{sc(G - f) : f \text{ is a cut-edge of } G\}$. In other words, e is a cut-edge such that one of the components of $G - e$ has minimum size among all subgraphs of G which can be separated by removing a cut-edge of G . Let G_1 and G_2 be the two components of $G - e$, where $v_1 \in G_1$, $v_2 \in G_2$, and $|V(G_1)| \leq |V(G_2)|$. For $i \in \{1, 2\}$, let $S_i = V(G_i) \setminus \{v_i\}$ and $s_i = |S_i|$. By the degree-sum formula, $ds_i + (d - 1) = \sum_{v \in V(G_i)} d_{G_i}(v) = 2|E(G_i)|$, whence it follows that $d(s_i + 1)$ is odd. Thus, both d and $s_i + 1$ are odd, and hence n is even; moreover, $s_i \geq 2$, and hence $n \geq 6$. See Figure 3 for an illustration.



Figure 3: A d -regular multigraph with $\kappa'(G) = 1$.

We now consider three cases based on the cardinality of $s_1 + 1$.

Case 1: $s_1 + 1 = 3$. In this case, the structure of the graph is determined uniquely, and the vertex partition $\{S_1, \{v_1\}, \{v_2\}, S_2\}$ corresponds to the quotient matrix Q defined in the statement of the Theorem; see Figure 4 for an illustration. Therefore, the inequality $\lambda_2(G) \geq \rho(d, n)$ holds for all d and n .

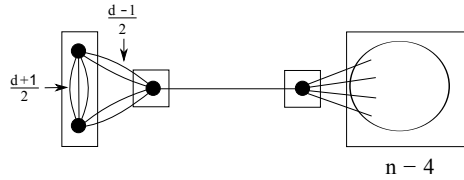


Figure 4: A d -regular multigraph with $\kappa'(G) = 1$ and $s_1 = 2$.

Case 2: $s_1 + 1 = 5$. Consider the partition $\{S_1, \{v_1\}, \{v_2\}, S_2\}$ and the corresponding quotient matrix:

$$Q' = \begin{pmatrix} d - \frac{d-1}{4} & \frac{d-1}{4} & 0 & 0 \\ d-1 & 0 & 1 & 0 \\ 0 & 1 & 0 & d-1 \\ 0 & 0 & \frac{d-1}{n-6} & d - \frac{d-1}{n-6} \end{pmatrix}.$$

Let $\rho'(d, n) = \lambda_2(Q')$. By Corollary 2.2, $\lambda_2(G) \geq \lambda_2(Q') = \rho'(d, n)$. Note that d is odd, and that due to the partition structure, $n \geq 10$. Thus, to show that $\lambda_2(G) \geq \rho(d, n)$ holds for all d and n , we will show that $\lambda_2(G) \geq \rho(d, n)$ holds when $d = 3$ and $n \in \{10, 12\}$, and that

$$\rho'(d, n) \geq \rho(d, n) \tag{3}$$

holds for all other values of d and n . To verify that $\lambda_2(G) \geq \rho(d, n)$ holds when $d = 3$ and $n \in \{10, 12\}$, we compute the second-largest eigenvalues of all possible multigraphs which have these parameters, and compare them to $\rho(3, 10)$ and $\rho(3, 12)$, respectively; the enumeration procedure is described in the Appendix. For all other values of d and n , we verify (3) by separating it into the following cases and using symbolic computation in Mathematica; see the Appendix for details. See also Case 3 below for a more detailed explanation of why this computation is sufficient to establish the claim.

- a) $d = 3, n \geq 14$. Fix $d = 3$ and $x = \frac{1689}{600}$. Then, $\det(xI - Q') > 0$ and $\det(xI - Q) < 0$ hold for all $n \geq 14$.
- b) $d = 5, n \in \{10, 12\}$. Fix $d = 5$ and $x = \frac{47}{10}$. Then, $\det(xI - Q') > 0$ and $\det(xI - Q) < 0$ hold for $n = 10$ and $n = 12$.
- c) $d = 7, n = 10$. Fix $d = 7$ and $x = \frac{333}{50}$. Then, $\det(xI - Q') > 0$ and $\det(xI - Q) < 0$ hold for $n = 10$.
- d) $d = 5, n \geq 14$; $d = 7, n \geq 12$; $d \geq 9, n \geq 10$. Fix $x = d - \frac{1}{5} - \frac{1}{n-5}$. Then, $\det(xI - Q') > 0$ and $\det(xI - Q) < 0$ hold for all values of d and n described in this case.

Case 3: $s_1 + 1 \geq 7$. In this case, we consider the vertex partition of G with the sets $S_1 \cup \{v_1\}$ and $S_2 \cup \{v_2\}$; see Figure 5 for an illustration.

The second-largest eigenvalue of the quotient matrix Q'' corresponding to this vertex partition is equal to $d - \frac{1}{s_1+1} - \frac{1}{s_2+1}$. By Corollary 2.2,

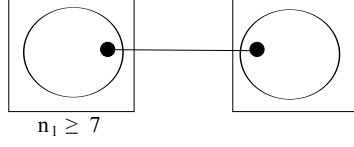


Figure 5: Partition of $V(G)$ into $S_1 \cup \{v_1\}$ and $S_2 \cup \{v_2\}$.

$\lambda_2(G) \geq \lambda_2(Q'') = d - \frac{1}{s_1+1} - \frac{1}{s_2+1} \geq d - \frac{1}{7} - \frac{1}{n-7}$, where the last inequality follows from the fact that $s_2 + 1 \geq s_1 + 1 \geq 7$. Note that n is even, d is odd, $d \geq 3$, and due to the partition structure, $n \geq 14$. Thus, to show that $\lambda_2(G) \geq \rho(d, n)$ holds for all d and n , we will show that $\lambda_2(G) \geq \rho(d, n)$ holds when $d = 3$ and $n \in \{14, 16, 18\}$, and that

$$d - \frac{1}{7} - \frac{1}{n-7} \geq \rho(d, n) \quad (4)$$

holds for all other values of d and n . To verify that $\lambda_2(G) \geq \rho(d, n)$ holds when $d = 3$ and $n \in \{14, 16, 18\}$, we compute the second-largest eigenvalues of all possible multigraphs which have these parameters, and compare them to $\rho(3, 14)$, $\rho(3, 16)$, and $\rho(3, 18)$, respectively; the enumeration procedure is described in the Appendix. For all other values of d and n , we verify (4) as follows.

Note that $\det(xI - Q)$ is a monic polynomial of degree 4, with roots $\lambda_1(Q)$, $\lambda_2(Q)$, $\lambda_3(Q)$, and $\lambda_4(Q)$; all roots are real, since they interlace the eigenvalues of G . Moreover, $\lambda_1(Q) + \lambda_2(Q) + \lambda_3(Q) + \lambda_4(Q) = \text{trace}(Q) = d + \frac{d+1}{2} - \frac{d-1}{n-4}$ and $\lambda_1(Q) = d$, which implies that $\lambda_2(Q) + \lambda_3(Q) + \lambda_4(Q) = \frac{d+1}{2} - \frac{d-1}{n-4}$. By Theorem 4.1, $\lambda_2(Q) \geq d - \frac{1}{3} - \frac{1}{n-3}$; thus, $\lambda_3(Q) + \lambda_4(Q) < 0$ for all $d \geq 3$ and $n \geq 14$. Since $\lambda_4(Q) \leq \lambda_3(Q)$, it follows that $\lambda_4(Q) < 0$. Finally, note that

$$\lambda_4(Q) < 0 < d - \frac{1}{3} - \frac{1}{n-3} < d - \frac{1}{7} - \frac{1}{n-7} < d = \lambda_1(Q).$$

Thus, showing that (4) holds is equivalent to showing that

- a) $\det(xI - Q) > 0$ for $x = d - \frac{1}{3} - \frac{1}{n-3}$, and
- b) $\det(xI - Q) < 0$ for $x = d - \frac{1}{7} - \frac{1}{n-7}$,

whence it follows that $\lambda_3(Q) \leq d - \frac{1}{3} - \frac{1}{n-3} \leq \lambda_2(Q) \leq d - \frac{1}{7} - \frac{1}{n-7} \leq \lambda_1(Q)$. Using symbolic computation in Mathematica, we can verify that a) holds for all $d \geq 3$ and $n \geq 14$, and b) holds when $d = 3$ and $n \geq 20$, and when $d \geq 5$ and $n \geq 14$; for details, see the Appendix. Since the case $d = 3, n \in \{14, 16, 18\}$ was verified by enumeration, this completes the proof. \square

Observation 4.3. *When $t = 1$, Theorem 4.1 states that if $\lambda_2(G) < d - \frac{1}{3} - \frac{1}{n-3}$, then $\kappa'(G) \geq 2$. Case 3 of the proof of Theorem 4.2 guarantees that $\rho(d, n) > d - 1/3 - 1/(n-3)$, which means that $\rho(d, n)$ is a better bound than $d - 1/3 - 1/(n-3)$.*

Observation 4.4. *Let G be the d -regular multigraph on 6 vertices with $d \geq 3$ and $\kappa'(G) = 1$. Then $\lambda_2(G) = \frac{1}{4}(d - 1 + \sqrt{9d^2 - 10d + 17}) = \rho(d, 6)$, where $\rho(d, n)$ is defined as in the statement of Theorem 4.2. Thus, the bound in Theorem 4.2 is the best possible for this infinite family of multigraphs.*

5 Conclusion

In this paper, we presented upper bounds for the second-largest eigenvalues of regular graphs and multigraphs, which guarantee a desired vertex- or edge-connectivity. The given bounds improve on several previous results, and hold with equality for infinite families of graphs. In deriving these bounds, we used computer-aided symbolic algebra, which synergizes well with the technique of eigenvalue interlacing; this combination gives a viable approach to investigating spectral bounds guaranteeing graph theoretic properties, which differs from the typical analytic strategies used in similar derivations. As part of our proof of Theorem 4.2, we developed an approach to enumerate all multigraphs with certain properties, by adding edges to certain simple graphs. Since the theory and available software for enumerating simple graphs are generally better-developed than their analogues for multigraphs, it may be worth investigating this strategy further and adapting it to other applications. Another problem of interest is to obtain bounds on the second-largest eigenvalues of a graph which guarantee a desired connectivity, and depend on other graph invariants such as girth or circuit rank. In particular, one could explore the utility of symbolic computer algebra used together with eigenvalue interlating in deriving such bounds.

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References

- [1] N.M.M. de Abreu, Old and new results on algebraic connectivity of graphs. *Linear Algebra and its Applications*, 423: 5373 (2007).
- [2] A.E. Brouwer and W.H. Haemers, *Spectra of Graphs*, Springer, New York, 2011.
- [3] S.L. Chandran, Minimum cuts, girth and spectral threshold. *Information Processing Letters*, 89(3): 105–110 (2004).
- [4] J. Chen and I. Safro, Algebraic distance on graphs. *SIAM Journal on Scientific Computing*, 33(6): 3468–3490 (2011).
- [5] S.M. Cioaba, Eigenvalues and edge-connectivity of regular graphs. *Linear Algebra and its Applications*, 432(1): 458–470 (2010).
- [6] M. Fiedler, Algebraic connectivity of graphs. *Czechoslovak Mathematical Journal*, 23(98): 298–305 (1973).
- [7] A. Frank, Augmenting graphs to meet edge-connectivity requirements. *SIAM Journal on Discrete Mathematics*, 5(1): 25–53 (1992).
- [8] S.L. Hakimi, On the realizability of a set of integers as degrees of the vertices of a simple graph. *SIAM Journal on Applied Mathematics*, 10: 496–506 (1962).
- [9] P. Hanlon, Enumeration of graphs by degree sequence. *Journal of Graph Theory*, 3(3): 295–299 (1979).

- [10] S. Hoory, N. Linial, and A. Wigderson, Expander graphs and their applications. *Bulletin of the American Mathematical Society*, 43: 439–561 (2006).
- [11] S. Hu and L. Qi, Algebraic connectivity of an even uniform hypergraph. *Journal of Combinatorial Optimization*, 24(4): 564–579 (2012).
- [12] A. Jamakovic and P. Van Mieghem, On the robustness of complex networks by using the algebraic connectivity. *International Conference on Research in Networking*, Springer Berlin Heidelberg, 2008.
- [13] A. Jamakovic and S. Uhlig, On the relationship between the algebraic connectivity and graph’s robustness to node and link failures. *3rd EuroNGI Conference on Next Generation Internet Networks*, IEEE, 2007.
- [14] M.-Y. Kao, Data security equals graph connectivity. *SIAM Journal on Discrete Mathematics*, 9(1): 87–100 (1996).
- [15] S. Kirkland, J.J. Moliterno, M. Neumann, and B.L. Shader, On graphs with equal algebraic and vertex connectivity. *Linear Algebra and its Applications*, 341: 45–56 (2002).
- [16] M. Krivelevich and B. Sudakov, Pseudo-random graphs. *More Sets, Graphs and Numbers*, 15: 199–262 (2006).
- [17] W. Liu, H. Sirisena, K. Pawlikowski, and A. McInnes, Utility of algebraic connectivity metric in topology design of survivable networks. *7th International Workshop on Design of Reliable Communication Networks*, IEEE pp. 131–138, 2009.
- [18] S. O, Algebraic Connectivity of Multigraphs. *arXiv:1603.03960* (2016).
- [19] S. O, The vertex-connectivity and second largest eigenvalue in regular multigraphs. *Linear Algebra and its Applications*, 491: 4–14 (2016).
- [20] A.A. Rad, M. Jalili, and M. Hasler, A lower bound for algebraic connectivity based on the connection-graph-stability method. *Linear Algebra and its Applications*, 435(1), 186–192 (2011).
- [21] F. Ruskey, R. Cohen, P. Eades, and A. Scott, Alley CATs in search of good homes. *Congressus Numerantium*, 97–110 (1994).

- [22] J.P.G. Sterbenz, D. Hutchison, E.K. Çetinkaya, A. Jabbar, J.P. Rohrer, M. Schöller, and P. Smith, Resilience and survivability in communication networks: Strategies, principles, and survey of disciplines. *Computer Networks*, 54: 1245–1265 (2010).
- [23] M.S. Taqqu, and J.B. Goldberg, Regular multigraphs and their application to the Monte Carlo evaluation of moments of non-linear functions of Gaussian random variables. *Stochastic Processes and their Applications*, 13(2): 121–138 (1982).
- [24] D.B. West, *Introduction to Graph Theory*, Prentice Hall, Inc., Upper Saddle River, NJ, 2001.
- [25] C.W. Wu, Algebraic connectivity of directed graphs. *Linear and Multilinear Algebra*, 53(3): 203–223 (2005).

Appendix

Below we provide the Mathematica code used to calculate the minimum of the second root of the characteristic polynomial in the proof of Theorem 3.2, and to check some cases in the proof of Theorem 4.2. The version of Mathematica used is 10.0.0.0 for 64-bit Microsoft Windows. We also include additional details about the procedure of enumerating certain multigraphs in the proof of Theorem 4.2.

Theorem 3.2: symbolic reductions

```

secondroot = (1/2)(d - m1/s1 - m2/s2 + ((d - m1/s1 - m2/s2)^2
+ 4(m1^2/s1 + m2^2/s2 - (m1m2)/(s1s2))))^(1/2);
s2 = n - 1 - s1;
m1 = d - m2;
Reduce[D[secondroot, m2] == 0 && 2 ≤ s1 ≤ n - 3 && m2 > 0 && d ≥ 3, m2]

s1 ≥ 2 && n ≥ 3 + s1 && d ≥ 3 && m2 ==  $\frac{-d+dn-3ds1+2dns1-2ds1^2}{-1+n-4s1+4ns1-4s1^2}$ ;
m2 =  $\frac{-d+dn-3ds1+2dns1-2ds1^2}{-1+n-4s1+4ns1-4s1^2}$ ;
FullSimplify[secondroot && d ≥ 3]

d -  $\frac{dn}{n+4ns1-(1+2s1)^2}$  && d ≥ 3

```

$$\text{Minimize} \left[\left\{ d - \frac{dn}{n+4ns_1-(1+2s_1)^2}, n \geq 5, d \geq 3, 2 \leq s_1 \leq n-3 \right\}, s_1 \right]$$

$$\begin{cases} \frac{-25d+8dn}{-25+9n} & d \geq 3 \& n \geq 5 \\ \infty & \text{True} \end{cases}, \quad s_1 \rightarrow \begin{cases} \text{Indeterminate} & !(d \geq 3 \& n \geq 5) \\ \frac{1}{2} \left(-1 - \sqrt{(-5+n)^2} + n \right) & \text{True} \end{cases}$$

Note that since $n \geq 5$, the argmin of s_1 in the last output is equal to 2.

Theorem 4.2: symbolic reductions

$$Q = \{ \{(d+1)/2, (d-1)/2, 0, 0\}, \{d-1, 0, 1, 0\}, \{0, 1, 0, d-1\}, \\ \{0, 0, (d-1)/(n-4), d-(d-1)/(n-4)\} \};$$

$$Q_{\text{prim}} = \{ \{d-(d-1)/4, (d-1)/4, 0, 0\}, \{d-1, 0, 1, 0\}, \\ \{0, 1, 0, d-1\}, \{0, 0, (d-1)/(n-6), d-(d-1)/(n-6)\} \};$$

$$\text{poly}Q = \text{CharacteristicPolynomial}[Q, x];$$

$$\text{poly}Q_{\text{prim}} = \text{CharacteristicPolynomial}[Q_{\text{prim}}, x];$$

Case 2a: note that both inequalities hold for $d = 3$ and $n \geq 14$.

$$d = 3; x = 1689/600;$$

$$\text{Reduce}[\text{poly}Q_{\text{prim}} > 0 \& n \geq 10, \text{Integers}]$$

$$\text{Reduce}[\text{poly}Q < 0 \& n \geq 10, \text{Integers}]$$

$$n \in \text{Integers} \& n \geq 13$$

$$n \in \text{Integers} \& n \geq 10$$

Case 2b: note that both inequalities hold for $d = 5$ and $n \in \{10, 12\}$.

$$d = 5; x = 47/10;$$

$$\text{Reduce}[\text{poly}Q_{\text{prim}} > 0 \& n \geq 10, \text{Integers}]$$

$$\text{Reduce}[\text{poly}Q < 0 \& n \geq 10, \text{Integers}]$$

$$n \in \text{Integers} \& n \geq 10$$

$$n == 10 || n == 11 || n == 12 || n == 13 || n == 14 || n == 15 || n == 16 || n == 17$$

Case 2c: note that both inequalities hold for $d = 7$ and $n = 10$.

$$d = 7; x = \frac{333}{50};$$

$$\text{Reduce}[\text{poly}Q_{\text{prim}} > 0 \& n \geq 10, \text{Integers}]$$

$$\text{Reduce}[\text{poly}Q < 0 \& n \geq 10, \text{Integers}]$$

$$n \in \text{Integers} \& n \geq 10$$

$$n == 10 || n == 11 || n == 12 || n == 13 || n == 14 || n == 15$$

Case 2d: note that both inequalities hold for $d = 5$ and $n \geq 14$, $d = 7$ and $n \geq 12$, and $d \geq 9$ and $n \geq 10$.

$$\text{Clear}[d]; x = d - 1/5 - 1/(n-5);$$

Reduce[polyQ < 0 & n ≥ 10 & d ≥ 3, Integers]

Reduce[polyQprim > 0 & n ≥ 10 & d ≥ 3, Integers]

$(d|n) \in \text{Integers} \& \& ((d == 4 \& n \geq 21) \|(d == 5 \& n \geq 14) \|(d == 6 \& n \geq 12) \|($

$(d == 7 \& n \geq 11) \|(d \geq 8 \& n \geq 10))$

$(d|n) \in \text{Integers} \& \& n \geq 10 \& d \geq 3$

Case 3: note that both inequalities hold for $d = 3$ and $n \geq 20$, and $d \geq 5$ and $n \geq 14$. The case $d = 3$, $n \in \{14, 16, 18\}$ is verified by enumeration in the next section.

$x = d - 1/7 - 1/(n - 7);$

Reduce[polyQ < 0 & n ≥ 14 & d ≥ 3, Integers]

Clear[x]; x = d - 1/3 - 1/(n - 3);

Reduce[polyQ > 0 & n ≥ 14 & d ≥ 3, Integers]

$(d|n) \in \text{Integers} \& \& ((d == 3 \& n \geq 19) \|(d \geq 4 \& n \geq 14))$

$(d|n) \in \text{Integers} \& \& n \geq 14 \& d \geq 3$

Theorem 4.2: enumerating multigraphs

Let A_{10} and A_{12} respectively be the sets of 3-regular multigraphs of order 10 and 12 with edge-connectivity 1, such that the removal of any cut-edge of these graphs produces components of order at least 5. Let A_{14} , A_{16} , and A_{18} respectively be the sets of 3-regular multigraphs of order 14, 16, and 18 with edge-connectivity 1, such that the removal of any cut-edge of these graphs produces components of order at least 7. These constraints imply that a graph in A_{10} or A_{14} must have exactly one cut-edge, a graph in A_{12} or A_{16} can have one or two cut-edges, and a graph in A_{18} can have one, two, or three cut-edges.

For $i \in \{5, 7, 9, 11\}$, let B_i be the set of all connected multigraphs which have degree sequence $\{3^{i-1}, 2\}$ and have no cut-edges. For any graph $H \in B_i$, $i \in \{5, 7, 9, 11\}$, define $v_2(H)$ to be the degree 2 vertex of H . Let J_2 be the graph consisting of two vertices joined by a double edge, let J_4 be the graph obtained by joining two copies of J_2 by one edge, and let J'_4 be a complete graph on four vertices with one edge removed. For $J \in \{J_2, J_4, J'_4\}$, define $v_2(J)$ to be one of the degree 2 vertices of J , and $v'_2(J)$ to be the other degree 2 vertex of J . For any $i, j \in \{5, 7, 9, 11\}$, define $B_i \rightleftharpoons B_j$ to be the set $\{H \dot{\cup} H' + \{v_2(H), v_2(H')\} : H \in B_i, H' \in B_j\}$ (where $\dot{\cup}$ denotes disjoint union). For any $i, j \in \{5, 7, 9, 11\}$ and $J \in \{J_2, J_4, J'_4\}$, define $B_i \rightleftharpoons J \rightleftharpoons B_j$

to be the set $\{H \dot{\cup} H' \dot{\cup} J + \{\{v_2(H), v_2(J)\}, \{v_2(H'), v'_2(J)\}\} : H \in B_i, H' \in B_j\}$. In other words, “ $\dot{=}$ ” denotes the set obtained by joining all possible pairs of graphs from the indicated families by a cut-edge incident to their degree 2 vertices. With this in mind, it is easy to see that

$$\begin{aligned}
A_{10} &= B_5 \dot{=} B_5 \\
A_{12} &= (B_5 \dot{=} B_7) \cup (B_5 \dot{=} J_2 \dot{=} B_5) \\
A_{14} &= B_7 \dot{=} B_7 \\
A_{16} &= (B_7 \dot{=} B_9) \cup (B_7 \dot{=} J_2 \dot{=} B_7) \\
A_{18} &= (B_7 \dot{=} B_{11}) \cup (B_9 \dot{=} B_9) \cup (B_7 \dot{=} J_2 \dot{=} B_9) \cup \\
&\quad (B_7 \dot{=} J_4 \dot{=} B_7) \cup (B_7 \dot{=} J'_4 \dot{=} B_7).
\end{aligned}$$

See Figure 6 for an illustration of these constructions.

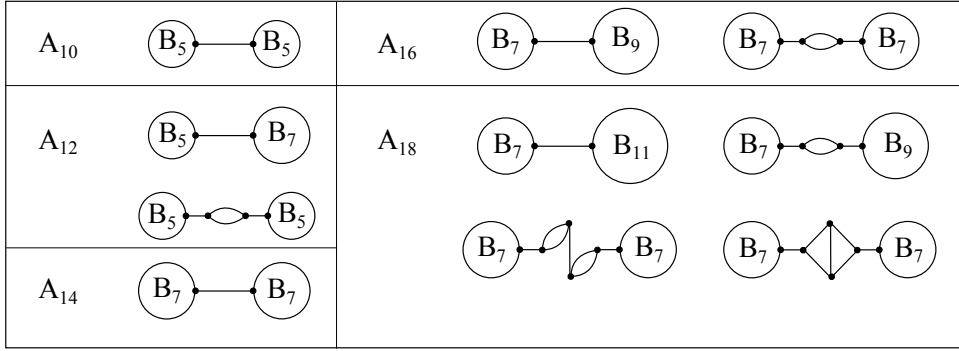


Figure 6: All possible 2-vertex-connected component and cut-edge structures of graphs in $A_i, i \in \{10, 12, 14, 16, 18\}$.

Thus, to find the graphs in $A_i, i \in \{10, 12, 14, 16, 18\}$, it suffices to find the graphs in $B_j, j \in \{5, 7, 9, 11\}$. Since the graphs in B_j are 3-regular and connected, they cannot have triple edges; moreover, they can have at most $\frac{j-1}{2}$ double edges. Let $M(\ell, j)$ be the set of multigraphs in B_j which have ℓ double edges. Then, $B_j = M(0, j) \cup \dots \cup M(\frac{j-1}{2}, j)$. We will now describe a procedure for enumerating the graphs in $M(\ell, j)$.

If the double edges of the graphs in $M(\ell, j)$ are replaced by single edges, the resulting graphs will be simple, 2-vertex-connected, and have degree sequence $\{3^{j-2\ell-1}, 2^{2\ell+1}\}$. There are well-known algorithms for generating all nonisomorphic simple graphs with a given degree sequence (cf. [8, 9, 21]); a

practical algorithm is implemented in the software system SageMath. Let $S(\ell, j)$ be the set of nonisomorphic simple graphs with degree sequence $\{3^{j-2\ell-1}, 2^{2\ell+1}\}$. Then, by adding double edges in all feasible ways to the simple graphs in $S(\ell, j)$, we can recover the multigraphs in $M(\ell, j)$. Specifically, a double edge can be added to a graph in $S(\ell, j)$ only where a single edge with two degree 2 endpoints already exists. Moreover, not every graph in $S(\ell, j)$ can have ℓ double edges added to it in a way that the resulting multigraph is in $M(\ell, j)$; similarly, it may be possible to add ℓ double edges to a graph in $S(\ell, j)$ in multiple ways so that the resulting multigraphs are in $M(\ell, j)$.

Let H be a graph in $S(\ell, j)$ and let $f(H)$ be the subgraph induced by the degree 2 vertices of H . Since the maximum degree of $f(H)$ is 2, $f(H)$ is the disjoint union of some paths and cycles. However, if $f(H)$ contains a cycle with less than j vertices, a multigraph in $M(\ell, j)$ cannot be obtained by doubling single edges of H with two degree 2 endpoints (since any resulting multigraph with degree sequence $\{3^{j-1}, 2\}$ will be disconnected). Similarly, if $f(H)$ contains more than one odd path, a multigraph in $M(\ell, j)$ cannot be obtained by doubling single edges of H with two degree 2 endpoints (since any resulting multigraph with degree sequence $\{3^{j-1}, 2\}$ will not have ℓ multiple edges).

Thus, let $S'(\ell, j) = \{H \in S(\ell, j) : f(H) \text{ is either a cycle } C_j, \text{ or contains exactly one odd path}\}$. For any graph H in $S'(\ell, j)$, the different maximum matchings (i.e. ℓ -matchings) of $f(H)$ correspond to different ways to add double edges to H . Let $F(H)$ be the set of multigraphs obtained by adding double edges to H corresponding to the different ℓ -matchings of $f(H)$. Then, $M(\ell, j) = \bigcup_{H \in S'(\ell, j)} F(H)$, $B_j = \bigcup_{\ell=0}^{(j-1)/2} M(\ell, j)$, and A_i can be obtained by joining pairs of graphs in B_j as described earlier. Note that the set of distinct maximum matchings of a graph whose components are paths (one of which is odd) can be found in linear time.

See Figure 7 for an illustration of this enumeration for $M(2, 7)$; the other sets of multigraphs $M(\ell, j)$ are handled analogously, and combined to obtain the graphs in A_i . Finally, for each multigraph in A_i , we can easily compute and compare the second-largest eigenvalue to $\rho(3, i)$; we have found that all of these eigenvalues are greater than or equal to $\rho(3, i)$, as desired.

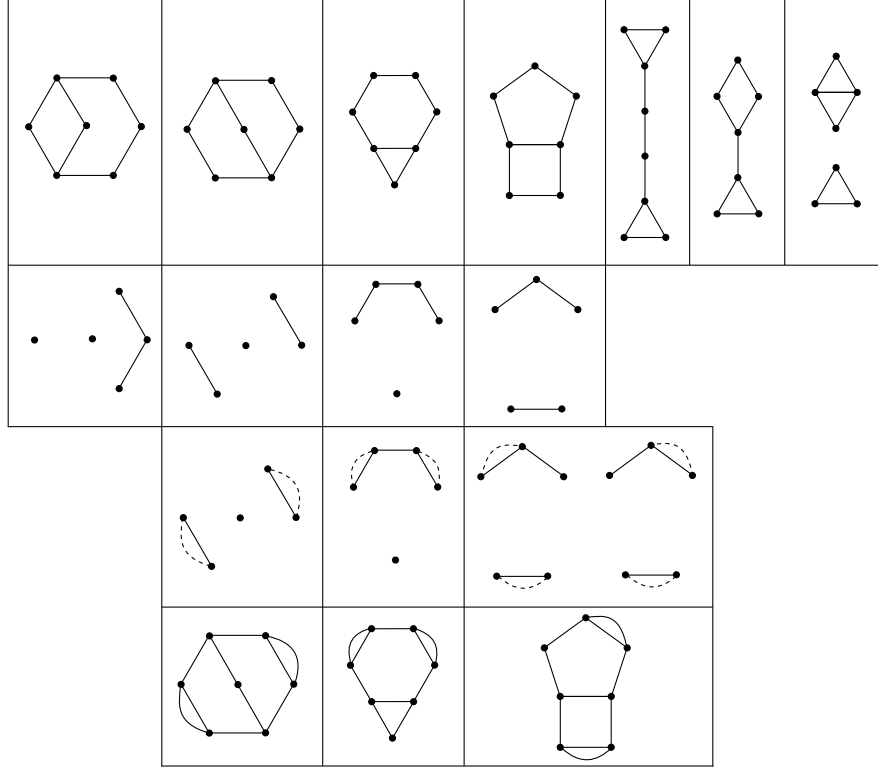


Figure 7: Enumerating the graphs in $M(2, 7)$. *Top row*: the graphs in $S(2, 7)$; the three graphs on the right are not 2-vertex-connected, so they are not considered further. *Second row*: $f(H)$ for the remaining graphs H ; the graph on the left has multiple odd paths, so it is not considered further. *Third row*: all possible 2-matchings of the remaining graphs in the second row. *Bottom row*: adding double edges specified by the matchings to obtain the graphs in $M(2, 7)$; the two matchings of the graph on the right happen to result in isomorphic multigraphs.